Investigation of Finite-Element Representations of the Geopotential

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As alternatives to lengthy series expansions for globally valid approximations of the earth's gravity field, piecewise approximation methods are evaluated. A single global equation is retained only for the point mass and second zonal terms of the geopotential; all finer structure undulations are modeled by a global family of locally valid functions. A degree 23 spherical harmonic series for the geopotential is replaced by finite-element approximations within the spherical shell out to 1.2 earth radii. This example application demonstrates conclusively the feasibility and desirability of the finite-element approach. An order of magnitude reduction in the calculation time for gravitational acceleration is realized over conventional calculations with spherical harmonic recursions.

Introduction

GRAVITATIONAL potential of an arbitrary body cannot be written as a closed expression. A number of truncated infinite series have been formulated[1-3] and truncations of such series currently serve as models of the gravity field for most computational purposes. These series are identical to, or are motivated by, classical product solutions of Laplace's equation in spherical coordinates.[1] One popular form of the spherical harmonic series for the geopotential at an arbitrary point (r = radius, ϕ = geocentric latitude, λ = east longitude) is

$$U=\frac{GM}{r}\sum_{n=0}^{N}\sum_{m=0}^{n}\left(\frac{R}{r}\right)^{n}P_n^m(\sin\phi)\left[C_n^m\cos m\lambda+S_n^m\sin m\lambda\right] \tag{1}$$

where R = a reference value of the earth's radius = 6378.160 km, GM = the earth's gravitational-mass constant = 398601.2 km^3/sec^2, $P_n^m(X)$ = associated legendre functions (non-normalized), and C_n^m, S_n^m = gravity coefficients determined to render simulated satellite motion in best agreement with observations.

Considerable research[2,3] has been directed toward developing stable recursion algorithms to compute Eq. (1) and the south, east, and radial acceleration components as

$$G_S=-\frac{1}{r}\frac{\partial U}{\partial\phi}$$

$$G_E=\frac{1}{r\cos\phi}\frac{\partial U}{\partial\lambda}$$

$$G_R=\frac{\partial U}{\partial r} \tag{2}$$

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In spite of the success achieved in developing feasible algorithms based upon Eqs. (1) and (2) and analogous series, the increasing burden of computing acceleration from ever more lengthy global gravity models consumes an ever larger fraction of the central processor time in trajectory/orbit integrations. This fact, and other considerations have motivated research[4,5] into other possible global gravity representations.

The present paper departs from current practice by separating from the onset the two important questions: 1) What gravity representation should be fit to observed data to estimate unknown model constants from satellite observations and/or surface gravimetry (and thereby establish a global gravity field approximation)? 2) In highly repetitive gravitational calculations (e.g., computing acceleration for use in integration of satellite orbits/missile trajectories), what gravity model is most efficient computationally?

In gravitational modeling research to date, both questions have been (perhaps subconsciously) addressed simultaneously. Explicit separation of the "optimum determination" and "optimum use" questions appears to be a most important consideration in further refinement of existing gravity models to accommodate the ever more precise and abundant observed data. Without engaging in the important quest for a model which can be best determined from observed data, we address the problem of determining an "optimum use" gravitational model.

Finite-Element Approach

The dominate global features of the gavity field are efficiently represented by the dominant low-degree terms in Eq. (1). This fact motivates the segregation of the total gravitational potential at a point (r, ϕ, λ) as

$$U=U_{\mathrm{REF}}(r,\phi,\lambda)+\Delta U(r,\phi,\lambda) \tag{3}$$

where it is understood that a single low-degree truncation of Eq. (1) is adopted to define U_{REF} globally. But instead of attempting to model "everything else" as a single global series, we shall determine a global family of locally valid disturbance functions to model ΔU and its gradient. Since the local equations must model only the gravity undulations (in addition to U_{REF}) in a specific local volume, it is reasonable to anticipate significantly more compact expressions than a single global expansion of comparable local accuracy. This is, of course, the thesis of the finite-element approach.

Having decided to pursue the finite-element approach, it is necessary to make several important decisions; it is necessary to define 1) what portion of the geopotential is to be approximated as ΔU in Eq. (3), 2) the specific mathematical structure to use as local approximations of ΔU and its gradient, 3) the size of the finite elements, 4) the procedure to be used in determining numerical values for coefficients in the local approximations, 5) the order of continuity requirements between adjacent approximations across their mutual boundaries of validity, and 6) procedures to evaluate the validity of the finite-element model.

These decisions are coupled and affect the accuracy and efficiency of the resulting finite-element model in a complicated way. Analytical attempts to resolve these issues a priori proved unsuccessful. Therefore, guided by intuition, we have developed experimental software and, based upon systematically collected empirical data, we arrived at a prototype finite-element model of the geopotential. We present several approaches for constructing gravitational finite-element models and summarize numerical experiments with them in the following.

Weighting Function Approach to Piecewise Continuous Approximation

Recent papers[6-10] document the theoretical development and application of a versatile piecewise continuous approximation technique. This weighting function approach determines an arbitrarily large family of locally valid functions which join with rigorous piecewise continuity through any prescribed order of partial differentiation. As applied to the approximation of a three variable function, $F(X_1, X_2, X_3)$ each final local approximation $\bar{F}(X_1, X_2, X_3)$ is defined as a weighted average of eight preliminary local approximations $\{f_{ijk}(X_1, X_2, X_3); \ i, j, k=0,1\}$ as

$$\bar{F}(X_1, X_2, X_3) = \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{k=0}^{1} W_{ijk}(X_1, X_2, X_3) \cdot f_{ijk}(X_1, X_2, X_3) \tag{4}$$

The eight preliminary approximation may have aribitrary mathematical structure (a most flexible and powerful feature of this approach) but must be determined in such a fashion that their respective centroids of validity lie at the eight vertices of a parallelopiped (in X_1, X_2, X_3, *space*), *which defines the volume in which the final local approximation* $\bar{F}(X_1, X_2, X_3)$ is valid. The form of the weight functions can be selected[6,7] to guarantee that $\bar{F}$ and its similarily determined adjacent approximations are continuous across their mutual boundaries of validity; the order of partial differentiation to which continuity is desired is controlled by selecting appropriate weight functions. Specifically, Table 1 gives the weight function $W_{111}(X_1, X_2, X_3)$ for various orders of continuity.† The remaining seven weight functions are obtained by reflecting $W_{111}(X_1, X_2, X_3)$ as

$$W_{000}(X_1, X_2, X_3) = W_{111}(1-X_1, 1-X_2, 1-X_3) \tag{5a}$$

$$W_{001}(X_1, X_2, X_3) = W_{111}(1-X_1, 1-X_2, \quad X_3) \tag{5b}$$

$$W_{010}(X_1, X_2, X_3) = W_{111}(1-X_1 \quad X_2, 1-X_3) \tag{5c}$$

$$W_{100}(X_k, X_2, X_3) = W_{111}(\quad X_1, 1-X_2, 1-X_3) \tag{5d}$$

$$W_{011}(X_1, X_2, X_3) = W_{111}(1-X_1, X_2, \quad X_3) \tag{5e}$$

$$W_{101}(X_1, X_2, X_3) = W_{111}(\quad X_1, 1-X_2, \quad X_3) \tag{5f}$$

$$W_{110}(X_1, X_2, X_3) = W_{111}(\quad X_1, \quad X_2, 1-X_3) \tag{5g}$$

The weight functions are positive and satisfy the constraint that

$$\sum_{i=0}^{1} \sum_{j=o}^{1} \sum_{k=o}^{1} W_{ijk}(X_1, X_2, X_3) = 1 \tag{6}$$

†Under the assumption that X_1, X_2, X_3 are nondimensional local coordinates with $(X_1=i, X_2=j, X_3=k)$ locating the eight vertices of the unit cube of validity of $F(X_1, X_2, X_3)$.

Table 1 Weight function W_{111} for various orders of continuity

Order of continuity m	$W_{111}(X_1, X_2, X_3,)$
0	$X_1X_2X_3$
1	$[X_1^2(3-2X_1)][X_2^2(3-2X_2)][X_3^2(3-2X_3)]$
2	$[X_1^3(10-15X_1+6X_1^2)][X_2^3(10 -15X_2+6X_2^2)][X_3^3(10-15X_3+6X_2^2)]$
	$\prod_{i=1}^{3} X_i^4(35\text{-}84X_i+70X_i^2-20X_i^3)]$
	$\prod_{i=1}^{3} X_i^4(35\text{-}84X_i+70X_i^2-20X_i^3)]$
. . .	. . .
m	$\prod_{i=1}^{3}\left\{\frac{(2m+1)!(-1)^m}{(m!)^2} \sum_{r=0}^{m} \frac{(-1)^r \binom{m}{r}}{2m-r+1} X_i^{2m-r+1}\right\}$

They may be interpreted geometrically as follows: The maximum value W_{ijk} is unity which occurs at $X_1=i$, $X_2=j$, $X_3=k$; the surfaces of constant weight are spherical in the vicinity of (i,j,k) but become increasingly angular until the surface of zero weight is the walls of the cube opposite to i,j,k. Thus, in Eq. (4), W_{ijk} causes the preliminary local approximation f_{ijk} to dominate $\bar{F}$ in the vicinity of f_{ijk}'s centroid of validity, but has no effect on $\bar{F}$ (in value or first m partial derivatives) along the opposite cell boundary. The key to piecewise continuity is the fact that $\bar{F}$ is completely defined (on each of the six boundary "walls") by the four preliminary approximations whose centroids are the vertices of the respective walls. Since the four preliminary approximations (defining any given cell wall) are shared by adjacent final approximations, it is clear that piecewise continuity is assured.

Local Gravity Approximations Via Taylor's Series

Given an apriori-determined global gravity model (e.g., a spherical harmonic series), perhaps the most obvious strategy of generating local approximations is by Taylor's series. The disturbance potential and its gradient can be locally approximated as the truncated Taylor's series of three variables

$$\Delta G_{ijk}(r,\phi,\lambda) \cong \Delta G(r_i,\phi_j,\lambda_k) + \sum_{n=1}^{M} \sum_{I=1}^{n} \sum_{J=1}^{n-1} G_{IJK} X_1^I X_2^J X_3^K \tag{7}$$

where

$$\Delta G_{ijk} \equiv \begin{Bmatrix} \Delta U \\ -1/r(\partial(\Delta U)/\partial \ \phi) \\ 1/r\cos\phi(\partial(\Delta U)/\partial \ \lambda) \\ \partial(\Delta U)/\partial r \end{Bmatrix} \equiv \begin{Bmatrix} \Delta U \\ \Delta G_S \\ \Delta G_E \\ \Delta G_R \end{Bmatrix} \tag{8}$$

$$G_{IJK} \equiv \frac{(\Delta r)^I (\Delta\phi)^J (\Delta\lambda)^K}{I!\ J!\ K!} \frac{\partial^n}{\partial r^I \partial\phi^J \partial\lambda^K} \begin{Bmatrix} \Delta U \\ \Delta G_S \\ \Delta G_E \\ \Delta G_R \end{Bmatrix}_{(r,\phi,\lambda)=(r_i,\phi_j,\lambda_k)} \tag{9}$$

$K=n-I$-J, $M=$ order of the local Taylor's series. (r_i, ϕ_j, λ_k) = an arbitrary local expansion point [which is the centroid of validity of Eq. (7)], $(2\Delta r,\ 2\Delta\phi,\ 2\Delta\lambda)$ = dimensions of the region of validity of Eq. (7), and $(X_1, X_2, X_3) \equiv ((r-r_i)/\Delta r,\ \ (\phi-\phi_j)/\Delta\phi,\ \ (\lambda-\lambda_k)/\Delta\lambda)$ = nondimensional local coordinates.

The partial derivatives Eq. (9) can be rigorously derived from the parent global model of ΔU. Reference 14 gives analytical expressions for computation of the partial derivatives specifically for the case that the parent global model of ΔU is a spherical harmonic series. It should be pointed out that specific numerical values for the elements of G_{IJK} are computed a priori and stored; for centroids of validity distributed over the (r,ϕ,λ) space according to some specified pattern.

In using (say) the foregoing weighting function formulation to compute local disturbance acceleration (from a piecewise continuous, finite-element model), the appropriate, previously computed, set of G_{IJK} coefficients are employed to compute Eqs. (7) as preliminary local aprroximations for substitution into

$$\begin{Bmatrix} U(r,\phi,\lambda) \\ G_S(r,\phi,\lambda) \\ G_E(r,\phi,\lambda) \\ G_R(r,\phi,\lambda) \end{Bmatrix} = \begin{Bmatrix} U_{\text{REF}}(r,\phi,\lambda) \\ -1/r(\partial U_{\text{REF}}/\partial\phi) \\ 1/r\cos\phi(\partial U_{\text{REF}}/\partial\lambda) \\ \partial U_{\text{REF}}/\partial r \end{Bmatrix} + \sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{1} W_{ijk}(X_1, X_2, X_3)\Delta G_{ijk}(r,\phi,\lambda) \tag{10}$$

where

$$X_1 = (r-r_0)/\Delta r,\ X_2 = (\phi-\phi_0)/\Delta\phi,\ X_3 = (\lambda-\lambda_0)/\Delta\lambda$$

for nondimensional local coordinates and (r_0,ϕ_0,λ_0) are coordinates of the "lower left corner" of Eq. (1)'s region of validity: $\{0 \le X_i \le 1;\ i=1,2,3\}$.

The gravity representation (10) leads to an nonuniform distribution of errors. Observe that the approximations become exact as the displacement of the evaluation point from a centroid of validity (expansion point) decreases to zero; but more generally, the final approximation is the average of eight approximations containing errors. This observation led us to expect from the onset that Taylor's series would not be the optimum choice of preliminary approximation functions; one should seek preliminary approximations with more uniform error distributions.

Local Gravity Approximations via Least Squares Approximation

As an alternative to the local Taylor's series [Eq. (7)], we consider as the local model of disturbance gravity

$$\begin{Bmatrix} \Delta U \\ \Delta G_S \\ \Delta G_E \\ \Delta G_R \end{Bmatrix} = \sum_{n=0}^{M}\sum_{I=0}^{n}\sum_{J=0}^{n-I} \begin{Bmatrix} U_{IJK} \\ S_{IJK} \\ E_{IJK} \\ R_{IJK} \end{Bmatrix} F_{IJK}(X_1,X_2,X_3) \tag{11}$$

where $K=n-I$-J.

$$[F_{000};F_{001},F_{010},F_{1000};F_{002},$$
$$F_{011},F_{101},F_{020},F_{110},F_{200};\ldots,F_{M00}(X_1,X_2,X_3)]$$

are a suitable set of linearly independent basis functions, and

$$\{U\}^T = [U_{000}U_{001}U_{010}U_{100}\ldots U_{M00}],$$
$$\{S\}^T = [S_{000}S_{001}S_{010}S_{100}\ldots S_{M00}],$$
$$\{E\}^T = [E_{000}E_{001}E_{010}E_{100}\ldots E_{M00}],$$
$$\{R\}^T = [R_{000}R_{001}R_{010}R_{100}\ldots R_{M00}],$$

are coefficients determined so that the sum square residual error (between Eq. (11) and the parent model of disturbance gravity) is minimized.

In particular, if the least square coefficient estimates are determined by fitting Eq. (11) to local evaluations of a global gravity model, then the coefficient estimates are given by the normal equations[11]

$$\{U\} = [B]\quad \{\Delta U_c\}$$
$$\{S\} = [B]\quad \{\Delta GS_c\}$$
$$\{E\} = [B]\quad \{\Delta GE_c\}$$
$$\{R\} = [B]\quad \{\Delta GR_c\} \tag{12}$$

where

$$[B] \equiv [(A^T WA)^{-1} A^T W] \tag{13}$$

$$[A] \equiv \begin{bmatrix} F_{000}(X_{11},X_{21},X_{31}) & F_{100}(X_{11},X_{21},X_{31}) & \cdots & F_{M00}(X_{11},X_{21},X_{31}) \\ F_{000}(X_{12},X_{22},X_{32}) & F_{100}(X_{12},X_{22},X_{32}) & \cdots & F_{M00}(X_{12},X_{22},X_{32}) \\ \vdots & \vdots & & \vdots \\ F_{000}(X_{1n},X_{2n},X_{3n}) & F_{100}(X_{1n},X_{2n},X_{3n}) & \cdots & F_{M00}(X_{1n},X_{2n},X_{3n}) \end{bmatrix} \tag{14}$$

$[W] = [w_{ij}]$ = an $n \times n$ positive definite weight matrix.

$$\left.\begin{aligned}
\{\Delta U_c\}^T &\equiv \{\Delta U(X_{11},X_{21},X_{31})\ \ \Delta U(X_{12},X_{22},X_{32})\ldots\Delta U(X_{1n},X_{2n},X_{3n})\}\\
\{\Delta GS_c\}^T &\equiv \{\Delta G_S(X_{11},X_{21},X_{31})\ \ \Delta G_S(X_{12},X_{22},X_{32})\ldots\Delta G_S(X_{1n},X_{2n},X_{3n})\}\\
\{\Delta GE_c\}^T &\equiv \{\Delta G_E(X_{11},X_{21},X_{31})\ \ \Delta G_E(X_{12},X_{22},X_{32})\ldots\Delta G_E(X_{1n},X_{2n},X_{3n})\}\\
\{\Delta GR_c\}^T &\equiv \{\Delta G_R(X_{11},X_{21},X_{31})\ \ \Delta G_R(X_{12},X_{22},X_{32})\ldots\Delta G_R(X_{1n},X_{2n},X_{3n})\}
\end{aligned}\right\} \quad (15)$$

are local evaluations of disturbance potential and acceleration (from a parent, global gravity model) at the set of points

$$(X_{1\ell},X_{2\ell},X_{3\ell}) = \left(\frac{r_\ell - r_i}{\Delta r},\frac{\phi_\ell - \phi_j}{\Delta\phi},\frac{\lambda_\ell - \lambda_k}{\Delta\lambda}\right);\ell = 1,2,\ldots,n \quad (16)$$

in the vicinity of the centroid of validity (r_i,ϕ_j,λ_k). It has been found advantageous to generate the data (15) on a uniform grid in the local $(X_{1\ell},X_{2\ell},X_{3\ell})$ space; providing this grid is held identical for all cells, then Eqs. (13) and (14) can be computed only once and simply reused in (12) to operate upon appropriate local data to generate the entire global set of local coefficients for use in Eq. (11).

An infinity of choices exist for the basis functions in (11); after some experimentation, we selected the set of all Chebyshev polynomial products up to Mth order (Table 2) as

$$\begin{Bmatrix} F_{000}\\ F_{001}\\ F_{010}\\ F_{100}\\ F_{002}\\ F_{011}\\ \cdot\\ \cdot\\ \cdot\\ F_{IJK}\\ \cdot\\ \cdot\\ \cdot\\ F_{M00}\end{Bmatrix} = \begin{Bmatrix} 1\\ T_1(X_3)\\ T_1(X_2)\\ T_1(X_1)\\ T_2(X_3)\\ T_1(X_2)T_1(X_3)\\ \cdot\\ \cdot\\ \cdot\\ T_I(X_1)T_J(X_2)T_K(X_3)\\ \cdot\\ \cdot\\ \cdot\\ T_M(X_1)\end{Bmatrix} \quad (17)$$

Numerical Tradeoff Studies

To gather the empirical evidence upon which to base selections from the many alternatives implicit in the above developments, a versatile computer software system‡ has been developed. Figure 1 summarizes a portion of a tradeoff study whose objective is to help decide whether Taylor's series or locally fit Chebyshev polynomials should be employed as local gravity approximations. The RMS of the acceleration error norm was determined by computing the variance of a directly evaluated acceleration error sample. In this case, the sample consisted of 216 uniformly spaced error calculation points over each finite element. Using the RMS acceleration error norm as an accuracy criterion (for a given order of the local approximation) it is clear that the locally fit Chebyshev polynomials are superior to equal order Taylor's series. The very nonuniform error distribution characteristic of truncated Taylor's series (zero error at expansion point, but rapidly degrading away from that point) is partially compensated for in the weighting function approach by the bell shaped weights and redundancy of the method. However, the RMS of the locally fit Chebyshev polynomials have been found consistently superior for low order approximations (≤ 4), usually by an order of magnitude. Consideration of other goodness of fit criteria (smallness of mean error, smallness of maximum error, near Gaussian residuals, etc.), support the choice of locally fit polynomials over local Taylor's series.

Based upon the numerical experiments done this far, the following tentative conclusions have been drawn:

1) Rigorous piecewise continuity of the local approximations, while desirable from conceptual and esthetic viewpoints, appears weakly justified using the small RMS error criterion. The weighted average of eight local ap-

‡The FORTRAN IV language has been used exclusively, all computation has been carried out using the University of Virginia's Control Data Corporation CDC 6400 computer.

Table 2 Chebyshev polynomials

n	$t_n(X)$	(dt_n/dX)
0	1	0
1	X	1
2	$2X^2-1$	$4X$
3	$4X^3-3X$	$12X^2-3$
.	.	.
.	.	.
.	.	.

Recursions:

$t_n(X) = 2X\ t_{n-1}(X)\ \ -t_{n-2}(X)\ \ \ -1 \leq x \leq 1,\ \ n \geq 2$

$(dt_n/dX) = 2t_{n-1}(X)\ \ +2X\dfrac{dt_{n-1}}{dX} - \dfrac{dt_{n-2}}{dX}\ \ \ n \geq 3$

Shifted Chebyshev polynomianls:

$T_n(Y) = t_n(2Y-1)\ \ \ \ 0 \leq Y \leq 1$

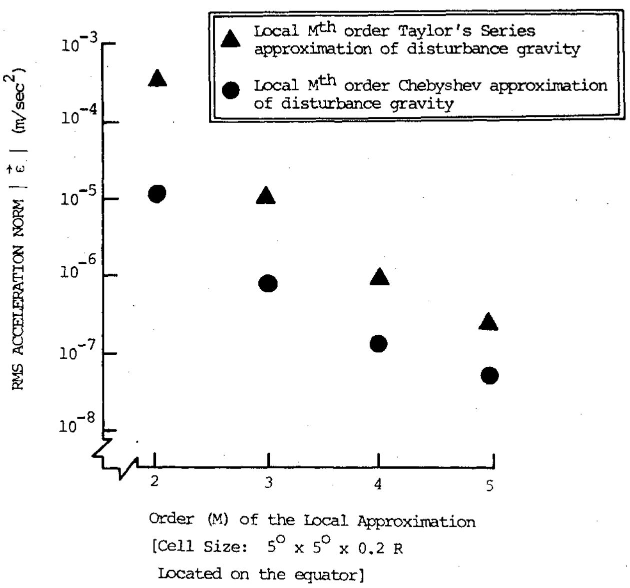

Fig. 1 Order of local approximations vs RMS acceleration approximation error. ϵ = (spherical harmonic acceleration model) (finite-element acceleration model).

proximations is consistently superior to any of the original approximations, but often not sufficiently superior to justify the 8:1 redundancy of the method.

2) By fixing the order of the local approximations and selecting a specific accuracy criterion, straightforward variation in the finite-element dimensions leads quickly to the family of maximum volume elements consistent with these two constraints. Global numerical tests support the conclusion that global decisions can usually be reliably made based upon local numerical experiments, so long as the local experiments include a full range of latitude variation.

3) Analysis of repeated statistical samples of acceleration errors, (between local Chebyshev approximations and a degree 23 global spherical harmonic series) taken from various finite elements support the following conclusions regarding distribution of approximation errors: a) The mean acceleration error has always been found to be at least one order of magnitude less than the sample RMS acceleration error, providing the sample was taken from at least 200 sample points located either on a uniform grid or randomly located within the element. b) The maximum acceleration error has always been found to be less than 4 times the sample RMS acceleration error, with the same sample restrictions as (3a).

Prototype Finite Element Model of the Geopotential

Based upon insight gained in parametric studies with the experimental software, a prototype finite-element model has been developed. The global finite model has the following structure

$$\begin{Bmatrix} U(r,\phi,\lambda) \\ G_S(r,\phi,\lambda) \\ G_E(r,\phi,\lambda) \\ G_R(r,\phi,\lambda) \end{Bmatrix} = \begin{Bmatrix} U(r,\phi\lambda) \\ G_S(r,\phi,\lambda) \\ G_E(r,\phi,\lambda) \\ G_R(r,\phi,\lambda) \end{Bmatrix}_{\mathrm{REF}} + \sum_{n=0}^{M}\sum_{I=0}^{n}\sum_{J=0}^{n-I} \begin{Bmatrix} U_{IJK} \\ S_{IJK} \\ E_{IJK} \\ R_{IJK} \end{Bmatrix} T_I(X_I)\,T_J(X_2)\,T_K(X_3) \qquad (18)$$

where

$$U_{\mathrm{REF}}(r,\phi,\lambda) \equiv \frac{GM}{r}\,[1+C_2^o\left(\frac{R}{r}\right)^2 P_2^o(\sin\phi)] \qquad (19)$$

$$G_{S_{\mathrm{REF}}} = -\frac{1}{r}\frac{\partial U_{\mathrm{REF}}}{\partial\phi} = \frac{-GM}{r^2}C_2^o\left(\frac{R}{r}\right)^2\frac{d}{d\phi}[P_2^o(\sin\phi)] \qquad (20)$$

$$G_{E_{\mathrm{REF}}} = \frac{1}{r\cos\phi}\frac{\partial U_{\mathrm{REF}}}{\partial\lambda} = 0 \qquad (21)$$

$$G_{R_{\mathrm{REF}}} = \frac{\partial U_{\mathrm{REF}}}{\partial r} = -\frac{GM}{r^2}\,[1+3C_2^o\left(\frac{R}{r}\right)^2 P_2^o(\sin\phi)] \qquad (22)$$

and U_{IJK}, S_{IJK}, E_{IJK}, R_{IJK} are the appropriate set of a priori computed local coefficients; computed according to Eqs. (12-14) with

$$F_{IJK}(X_1,X_2,X_3) = T_I(X_I) \quad T_J(X_2) \quad T_K(X_3) \qquad (23)$$

to accurately replace the spherical harmonic model[12] of disturbance gravity

$$\Delta U = \frac{GM}{r}\sum_{n=2}^{23}\sum_{m=1}^{n}\left(\frac{R}{r}\right)^n P_n^m(\sin\phi)\,[C_n^m\cos m\lambda + S_n^m\sin m\lambda] \qquad (24)$$

and its gradient.

Figure 2 is a projection of a global contour map of the radial disturbance acceleration $[\partial(\Delta U)/\partial r]$ on the earth's surface. To fully define the procedure for constructing the finite elements, the following decisions were made: 1) develop a finite-element model for the spherical shell within 1.2 earth radii; 2) adjust the finite-element size and/or the order of the local approximation so that the acceleration approximation errors enter (at worst) in the seventh significant figure; and 3) fix the order of the local approximations at $M=3$ [in Eq. (19)].

Holding the radial dimension fixed at 0.2 R and adjusting the longitude by latitude dimensions to maintain requirement (2) led to the set of 1500 finite-elements whose bases are shown in the flat projection of Fig. 3. The acceleration error residual norm

$$\Delta G = [\,(GSH - GFE)^T(GSH - GFE)\,]^{1/2} \qquad (25)$$

where

$$GSH = \begin{bmatrix} G_S \\ G_E \\ G_R \end{bmatrix}_{\text{Spherical harmonic series model}} \qquad (26)$$

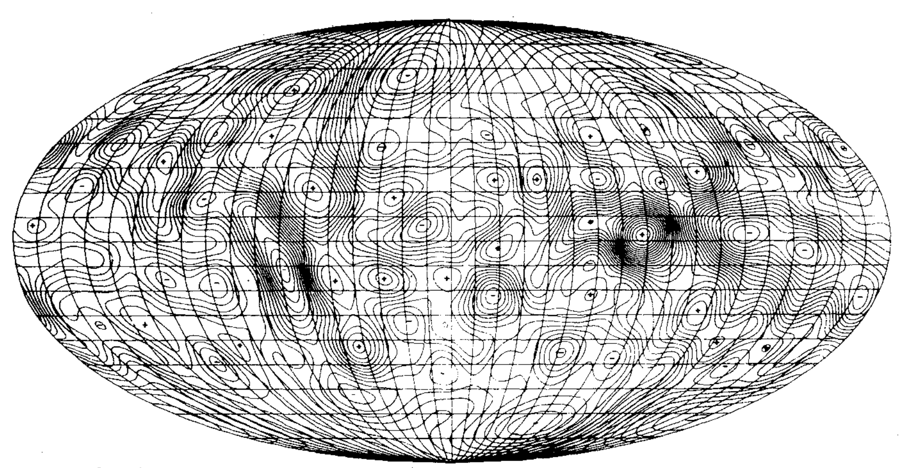

Fig. 2 Radial disturbance acceleration $\partial(\Delta U)/\partial r$ on the earth's surface (contour interval = 5×10^{-5} m/sec^2); at $[\lambda,\phi] \neq (79°, 2°)$, $\left.\frac{\partial(\Delta U)}{\partial r}\right|_{\max} = 1.03\times10^{-3}$ m/sec^2; at $[\lambda,\phi] \neq (121°, 3°)$ $\left.\frac{\partial(\Delta U)}{\partial r}\right|_{\min} = -0.66\times10^{-3}$ m/sec^2.

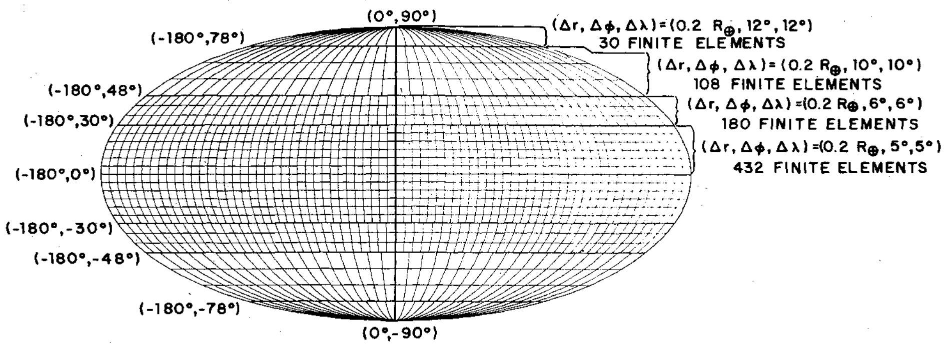

Fig. 3 Distribution of finite elements for a 1500 element 3rd order Chebychev model of the geopotential.

$$GFE = \begin{bmatrix} G_S \\ G_E \\ G_R \end{bmatrix}_{\text{finite-element model}} \tag{27}$$

was computed as were its RMS and maximum value for $N=23$ in Eqs. (1) and (2). We found that

$$\overline{\Delta G} = \frac{1}{N}\sum_{i=1}^{N} \Delta G_i \qquad = 0.000\,000\,03 \text{ m/sec}^2$$

$$\text{RMS}_{\Delta G} = \frac{1}{N}\left[\sum_{i=1}^{N} \Delta G_i{}^2\right]^{1/2} = 0.000\,002 \quad \text{m/sec}^2$$

$$\Delta G_{\text{MAX}} \qquad = 0.000\,008 \quad \text{m/sec}^2$$

$N=360\times 91=32760$ sample points ($1°\times 1°$ grid in the Northern hemisphere)

These errors are the worst case errors arising generally in the most anomalous region near the earth's surface; analogous statistical anslyses reveal the magnitude of the gravity modeling errors on the surface of a 1.03 R sphere are about one half of the above values while the errors are reduced by order of magnitude on the surface of the 1.2 R sphere.

This level of precision is probably satisfactory for integration of most missile trajectories and satellite orbits; typical single revolution position integration errors have been found to be on the order of 0-25 m. Clearly, greater precision can be easily achieved, if desired, by decreasing the finite-element size (thereby increasing the number of elements), or by increasing the order of local approximations (thereby decreasing the computational efficiency of the method).

The computational speed of the finite-element model versus typical spherical harmonic recursion[3] favors the finite-element model of Eq. (19) by about an order of magnitude The computational picture is complicated, however, by virtue of the fact that random access retrieval of previously stored coefficient subsets is necessary. In this case, each component of acceleration requires a total of (20)(1500) = 30,000 coefficients to define the entire global family of gravity functions (although only twenty are used in each element). Since the elements are large (hundreds of miles) compared to small errors (tens of miles) associated wirh two body or other simplified dynamic extrapolations, simple logic can be devised to bring into central memory several local sets of coefficients before they are needed and thereby hold the lost time during random access to a minimum. For the ballistic missile problem; most of the acceleration evaluations occur in two local regions (i.e., during atmospheric powered flight, and during re-entry); thus hundreds of acceleration evaluations are likely within a single finite element.

Conclusions

The finite-element approach to modeling the geopotential has been studied analytically and numerically. Many degrees of freedom exist in our approach to this problem. The specific finite-element model developed and discussed therein is not put forth as the optimal computational model of the geopotential. Rather, we believe the prototype finite model to be a representative finite-element geopotential model which will probably be improved upon in future refinements of our approach. The computational speed advantages over spherical harmonic expansions is clear, however (primarily because a 23rd order expansion is locally replaced by a 3rd order expansion). The ultimate computational speed advantage depends upon the number of random accesses required to maintain the appropriate local coefficients in core memory. For a given trajectory/orbit integration, simulations done to date support the conclusion that an order of magnitude savings is achieved. The finite-element software and coefficients are available upon request.

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